Indian J. Phys. **48**, 504–508 (1974)

An analytic representation for form factors

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(*Received* 7 *June* 1973)

An exponential formula with correct analytic property is suggested for form factors. The variable for exponentiation is obtained by mapping the entire plane of analyticity into the inside of a parabola. Some applications are discussed.

1. Introduction

A new method of analysis of experimental data has been proposed where conformal transformation is used to increase the rate of convergence of polynomial expansion of *amplitudes* having known analytic properties (Cutkosky & Deo 1968a, 1968b and Ciulli 1969). The increase is made optimum by mapping the whole domain of analyticity into the interior of a *figure of convergence*. The figure of convergence is obtained by finding the equipotential for the electrostatic problem in which the *physical* domain forms an earthed conductor in free space with a unit negative charge. In the $\cos\theta$-plane, (θ, being the c.m. scattering angle) the corresponding equipotentials are ellipses. An incomplete elliptic function (Cutokosky & Deo 1968a) of the first kind maps the cut $\cos\theta$-plane into the interior of an ellipse with focii at -1 and $+1$. The scattering amplitude or the scattering cross section can then be expanded in a rapidly convergent series in this 'elliptic' variable.

Form factors are simpler analytic functions having either a left or a right hand cut only. We shall denote this function of single variable by $f(t)$. Representations for form factors are usually (i) simple exponentials like $\exp(-at)$, (ii) poles at large mass values like $(M^2-t)^{-1}$ behaving like a heavy mass ghost propagator, (iii) resonance pole factors multiplying a suitable polynomial and (iv) the dipole fits like $(M^2-t)^{-2}$. Adequate attention has not been paid to the constraints imposed by physically measurable region for the form factors and the analytic requirements of the function. The analytic character depends on the microscopic structure which in its turn depends on the forces responsible for the physical phenomena for which the form factor is assumed to play the significant role. Here, we shall construct a suitable variable which incorporates the desired analyticity along the lines proposed by Deo & Parida (1971) and also suggest a simple exponential representation for the form factors. The related ideas will find applications in a large class of physical problems. Some relating to particle physics are indicated in this paper.

2. Analysis

Consider then the function $f(t)$ physically measurable from $t = 0$ to $t = \infty$ and that it is assumed to be analytic in the entire cut t plane, the cut extending from $t = -\infty$ to $t = -t_c$. The equipotentials for a semi-infinite line charge from 0 to ∞ are parabolas. According to the recipe stated above, the optimally mapped variable may be obtained by mapping the entire cut plane into the interior of a parabola. One such parabola was obtained (Cutkosky & Deo 1968b) by setting

$$\cos\theta = 1-(t/2q^2), \qquad \ldots \quad (1)$$

and then taking the limit $q^2 \to \infty$. In this limit the physical region extends from 0 to ∞ and the cut from $-\infty$ to some $-t_c$. It was shown that the mapped variable obtained from the incomplete elliptic function in this limit, $q^2 \to \infty$, was proportional to ($T = t/t_c$)

$$z(t) = (\log(\sqrt{T}+\sqrt{(T+1)}))^2 = (\sinh^{-1}\sqrt{T})^2. \qquad \ldots \quad (2)$$

A constant of proportionality can be always multiplied and is an adjustable parameter of the theory.

The formula, we suggest, is an exponential in this variale. We proceed to give some arguments for this. In case of finite interval like -1 to $+1$ in $\cos\theta$ plane, the system of orthogonal polynomials is closed so that one can analyse the experimental data by expanding in any of the classical polynomials, like Legendre or Gegenbauer or Jacobi polynomials or a convenient set of polynomials* weighted by the errors as was done by Cutkosky & Deo (1968a). In the present case, the interval is zero to infinity and there is only one class of classical polynomials, namely the Laguerre polynomials whose interval coincides with the physical region of interest here. It is easily shown that the Laguerre polynomials can be gotten by the mapping from the Jacobi polynomials because of the relation

$$P_n^{(a,b)}(\cos\theta) = P_n^{(a,b)}/(1-2t/4q^2) \xrightarrow[b\to\infty]{} L_n^{(a)}(t), \qquad \ldots \quad (3)$$

where $b = 4q^2$ and b tends to infinity.

As discussed by Deo *et al* (1971) it is also known Erdelyi (1953) that for Laguerre polynomials, the region of convergence is a parabola around the positive real axis with its focus at the origin. The mappings contained in eq. (2) maps whole of the cut plane of analyticity into the inside of such a parabola. The variable z has the correct analyticity built into it and it has been argued by Deo *et al* (1971) that an expansion in Laguerre polynomials with an exponential weight

*The Tshebyschef Polynomials are the best polynomials for expansion as they have unit norm on the mapped ellipse. This was pointed out recently to the author by Professor R.E. Cutkosky.

factor shall converge. The region of convergence, being not bounded, certain growth conditions have to be satisfied by the function to be expanded (Erdelyi 1953).

Relevant experimental results indicated that a simple exponential of the type

$$f(t) = f(0) \ \exp(-az) \qquad \ldots \quad (4)$$

is a fair fit to the data. We discuss below some of the features which are quite satisfactory and compare favourably with other representations of form factors.

3. Applications

(i) One useful application of our ideas may be illustrated by constructing an analytic form factor for the *Born term*. It is well known that the *Born cross section* is very high in meson physics and one has to consider an infinite number of ad-hoc substractions or completely ignore it with little or no reason. Consider for simplicity, pion exchange pole term $g^2/(\mu^2-t)$. The pion is on the mass shell, at $t = \mu^2$ and pion can be exchanged as a virtual particle for $t = -\infty$, to μ^2 and this could be considered as the *physical region* for the variable t. The cut, of course, extends from $t = 9\mu^2$ to ∞, and this is the only cut and there are no poles in the entire cut t plane.

So from the ideas outlined above, the parabolic variable is

$$z = (\sinh^{-1}\sqrt{(\mu^2-t)}/\sqrt{(8\mu^2)})^2, \qquad \ldots \quad (5)$$

and the 'Born term' with correct analytic form factor is

$$F_B = (g^2/\mu^2-t) \exp(-az). \qquad \ldots \quad (6)$$

A rough plot with Ferrari et al (1963) form factor shows that for a suitable a, $(a \sim 0.8)$ both form factors are very much the same for not too large values of $-t$.

(ii) The cut structure of the differential scattering cross section $d\sigma/d\Omega$ for high energy can be assumed to have a right cut only. One can represent with z of eq. (2)

$$s\frac{d\sigma}{d\Omega} = \left[s\frac{d\sigma}{d\Omega}\right]_{t=0} \exp(-az), \qquad \ldots \quad (7)$$

containing, of course, the unknown parameter a. For small t, $z(t) = =\alpha t+\beta t^2$, so that eq. (8) leads to the usual exp $(at+bt^2)$ form used for representing the high energy cross-section. Eq. (8) having the correct analytic structure should provide a much better fit to high energy data and hopefully, for some what higher values of t. It should, however, be remarked that in many strong interaction problems, the value of $t_c(= 4\mu^2)$ is too small and the asymptotic limit is reached at about 0.1 (Bev)^2, contrary to experimental observation near forward directions.

It is, always guessed that the nearby 2π-cut is too weak and the effective cut lies somewhere in the vector meson resonance region.

In this connection, we have shown that a Orear type fit for the differential cross section can also be obtained by suitable conformal mapping of the plane of analyticity and ignoring the inelastic cuts. Further, a constant slope factor for very large energy as observed in recent experiments (Beznogikh et al 1969 and Holder et al 1971) reflects the simple analytic structure at high energy and a form like eq. (7) would be anadequate representation of the data for small t.

(iii) The asymptotic limit of $f(t)$ is

$$f(t) \exp\left(-\tfrac{1}{2}(\log t)^2\right)$$

This asymptotic limit for form factors has been obtained by several authors from statistical consideration (Mack 1967, Jackiw 1969) Sudakov 1956, Cassandro et al 1964, Appilquist et al 1970) and also from eikonal approximations. It is remarkable that simple analytical considerations lead to results similar to those obtained by actual dynamical calculations.

(iv) The usefulness of the mapped variable $z(t)$ to approximate the left hand cut had also been pointed out by Cutkosky & Deo (1970) in their energy dependent analysis of $K^+ + p$ scattering in the low enegy region. It is hoped that the method can be used in formulating a better convergent expansion like the effective range theory and also in representing the differential scattering cross-section for non-forward angles.

(v) A very interesting case would be the form factor for systems having anomalous threshold. For a direct comparison with experiments we choose electromagnetic form factor of the deuteron where the anomalous threshold plays an important role. We write the form factor in the simple approximate form,

$$G_{Ed}(t) = \exp(-az(t))$$

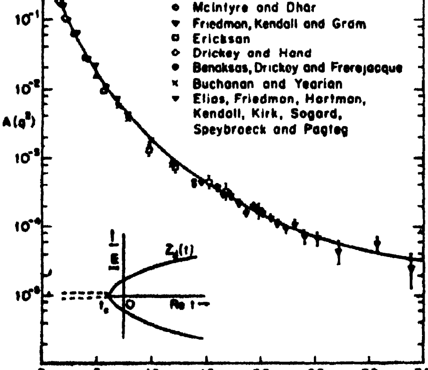

Fig. 1

The cut starts from the anomalous threshold $t = 16MB$, B being the binding energy of the deuteron and M the nucleon mass. The only unknown parameter is a. With $a = 1.64$, we obtain the excellent fit shown in figure 1. The data points are from tables and plot of Elias et al (1969). This may be considered an extremely sucessful application of the ideas of conformal mapping to incorporate analyticity in the presentation of experimental data.

Acknowledgment

I am grateful to Professor R. E. Cutkosky for having had discussions on the subject. I am also thankful to Mr. M. K. Parida for some help in calculations.

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